Boundary spectral behaviour for semiclassical operators in one dimension

Michael Hitrik

Department of Mathematics, University of California, Los Angeles, CA 90095-1555, USA
E-mail: hitrik@math.ucla.edu

Key words: spectrum, pseudospectrum, non-selfadjoint, Hamilton map

1991/2000 AMS Subject Classification: 34E05, 34L20, 47A12

Abstract: For a class of non-selfadjoint semiclassical operators in dimension one, we get a complete asymptotic description of all eigenvalues near a critical value of the leading symbol of the operator on the boundary of the pseudospectrum.

1 Introduction and main result

In his work [1], E. B. Davies considers the non-selfadjoint harmonic oscillator,

$$P(h) = h^2 D_x^2 + e^{i\alpha} x^2, \quad 0 < \alpha < \pi/2.$$

By an analytic continuation argument, it was established that the spectrum of $P(h)$ consists of the eigenvalues $\{e^{i\alpha/2} h(2k+1),\ k = 0,1,2,\ldots\}$, while the semiclassical pseudospectrum of $P(h)$, defined as the range of the symbol of $P(h)$, $p(x,\xi) = \xi^2 + e^{i\alpha}x^2$, on $\mathbf{R}^2$, is equal to the sector $\{z \in \mathbf{C}; 0 \leq \arg z \leq \alpha\}$. As $h \to 0$, the spectrum accumulates precisely at the origin, which is the only point z_0 at the boundary of the pseudospectrum, for which $dp = 0$ on $p^{-1}(z_0) \cap \mathbf{R}^2$. The angle at which the eigenvalues approach the origin is given here by $\alpha/2$. Now associated with the quadratic form p, there is a Hamilton map F, defined by

$$F = \frac{1}{2}\begin{pmatrix} p''_{x\xi} & p''_{\xi\xi} \\ -p''_{xx} & -p''_{x\xi} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -e^{i\alpha} & 0 \end{pmatrix},$$

with the eigenvalues $\pm\mu$, where $\mu = ie^{i\alpha/2}$. We may therefore reformulate the result of [1] by saying that the angle at which the spectrum approaches the boundary point 0 is given by $\arg(\mu/i)$. In this paper we shall give a straightforward extension of

this result to what we think is a natural class of non-selfadjoint h-pseudodifferential operators in dimension one. We shall show that, in the semiclassical limit, the direction of the spectrum near a critical value on the boundary of the pseudospectrum is determined by a suitable eigenvalue of the Hamilton map of the Hessian of the principal symbol at a corresponding critical point.

Without any restriction on the dimension, pseudospectra of non-selfadjoint semiclassical pseudodifferential operators have been studied in the recent paper [3], which extended some of the results of [2], [15], [16]. Following [16] and [3], let us consider the convection-diffusion operator with a quadratic potential,

$$P(x,hD_x)=(hD_x)^2+i(hD_x)+x^2=e^{x/2h}\circ\left((hD_x)^2+x^2+\frac{1}{4}\right)\circ e^{-x/2h}.$$

Using the last representation, one can prove that the spectrum of $P(x,hD_x)$ is given by $\{(2k+1)h+\frac{1}{4},\, k=0,1,2\ldots\}$, while the range of the symbol of $P(x,hD_x)$ fills out the region $\{z\in\mathbf{C};\mathrm{Re}\,z\geq(\mathrm{Im}\,z)^2\}$. In this case, the spectrum lies strictly inside the pseudospectrum, and Theorem 3 in [3] gives general conditions on a point z_0 at the boundary of the pseudospectrum of an operator $P=p^w(x,hD_x)+\mathcal{O}(h)$, to be away from the spectrum, as $h\to 0$. The first one is the principal type condition,

$$dp\neq 0\ \text{ along }\ p^{-1}(z_0),\tag{1.1}$$

and the second is the following dynamical hypothesis,

$$\text{For some }\lambda\in\mathbf{C},\text{ no complete trajectory of }H_{\mathrm{Re}(\lambda p)}\text{ is contained in }p^{-1}(z_0).\tag{1.2}$$

Asssuming (1.1) and (1.2) together with a basic analyticity assumption, it is established in [3] that there is a sufficiently small but fixed neighborhood of z_0 in $\mathbf{C}$ which does not intersect the spectrum of P. Back to the one-dimensional case, the purpose of this note is to describe the eigenvalue distribution near a boundary point z_0, in a simplest situation when the condition (1.1) fails to hold (so that (1.2) fails as well). In doing so, as in [3], we shall work under the analyticity assumptions, which will be essential for our methods.

Let

$$P=P^w(x,hD_x;h)$$

be the h–Weyl quantization of a symbol $P(x,\xi;h)$, which is a holomorphic function of (x,ξ) in a tubular neighborhood of $\mathbf{R}^2\subset\mathbf{C}^2$, with

$$P(x,\xi;h)=\mathcal{O}(1)m(\mathrm{Re}\,(x,\xi))\tag{1.3}$$

there. Here we assume that $1 \leq m \in C^\infty(\mathbf{R}^2)$ is an admissible weight function in the sense that for some fixed $C_0, N > 0$ we have

$$m(X) \leq C_0 \langle X - Y \rangle^N m(Y), \quad X, Y \in \mathbf{R}^2.$$

We next assume that as $h \to 0$, $P(x, \xi; h)$ has an expansion in the space of holomorphic functions satisfying the bound (1.3),

$$P(x, \xi; h) \sim \sum_{j=0}^{\infty} h^j p_j(x, \xi).$$

For $h > 0$ small enough, and when equipped with the domain $H(m)$, the naturally defined Sobolev space associated to the weight m, P becomes a closed densely defined operator on $L^2(\mathbf{R})$.

We shall denote the principal symbol of P by $p := p_0$, and make an additional assumption of the ellipticity of p near infinity,

$$|p(X)| \geq \frac{m(\operatorname{Re} X)}{C}, \quad |X| \geq C, \quad |\operatorname{Im} X| \leq \frac{1}{C}, \quad C > 0. \tag{1.4}$$

Throughout this paper, we shall work under the assumption that

$$m(X) \to \infty, \quad |X| \to \infty, \ X \in \mathbf{R}^2. \tag{1.5}$$

As before, associated to P, we introduce the semiclassical pseudospectrum

$$\Sigma(p) := p(\mathbf{R}^2). \tag{1.6}$$

This is a closed set, since p is proper, and we shall assume that $\Sigma(p)$ is not all of $\mathbf{C}$. It follows then from (1.5) that if z_0 is in the complement of $\Sigma(p)$, then $(P - z_0)^{-1}$ exists and is a compact operator on L^2. The analytic Fredholm theory implies that the spectrum of P is discrete and consists of eigenvalues of finite multiplicity.

Let $z_0 \in \partial\Sigma(p)$ and assume that $p^{-1}(z_0) \cap \mathbf{R}^2$ is a finite collection of points. It will in fact be sufficient to treat the case when the pre-image of z_0 is just a single point, say $(0, 0) \in \mathbf{R}^2$, and in what follows we shall work under this assumption. We introduce an exterior cone condition,

$$\begin{gathered} \text{There exist } \epsilon_0 > 0 \text{ and } \theta_0 \in \mathbf{R} \text{ such that} \\ \left(z_0 + (0, \epsilon_0) e^{i(\theta_0 - \epsilon_0, \theta_0 + \epsilon_0)}\right) \cap \Sigma(p) = \emptyset. \end{gathered} \tag{1.7}$$

Our final assumption is that $p - z_0$ vanishes at $(0, 0)$ precisely to the second order, so that in a neighborhood of this point, we have

$$|p(X) - z_0| \sim |X|^2.$$

The following is the main result of this work.

Theorem 1.1 *Assume that z_0 is at the boundary of the semiclassical pseudospectrum of P, defined in* (1.6)*, and that $p^{-1}(z_0) \cap \mathbf{R}^2 = (0,0)$, with $p - z_0$ vanishing there precisely to the second order. Furthermore, we make an exterior cone assumption* (1.7)*. Then there exists $\alpha \in \mathbf{C}$, $|\alpha| = 1$, such that $\mathrm{Re}\, \alpha p''(0,0) > 0$, and the spectrum of P in a sufficiently small but fixed neighborhood of z_0 in $\mathbf{C}$ is given by*

$$z_k = z_0 + G\left(h(k + \frac{1}{2}); h\right) + \mathcal{O}(h^\infty), \quad k \in \mathbf{N}.$$

Here $G(q;h)$ is holomorphic in $q \in \mathrm{neigh}(0, \mathbf{C})$, and has an asymptotic expansion in the space of such functions, as $h \to 0$,

$$G(q;h) \sim \sum_{j=0}^{\infty} h^j G_j(q).$$

We have $G_0(0) = 0$ and

$$\arg\left(\frac{\partial}{\partial q} G_0(0)\right) = \arg\left(\frac{\mu}{i}\right),$$

where μ is the unique eigenvalue of the Hamilton map of $p''(0,0)$, for which

$$\mathrm{Re}\,(\alpha\mu/i) > 0.$$

We illustrate Theorem 1.1 by comparing the results of a numerical computation of small eigenvalues of the operator $P = (hD_x)^2 + cx^2 + x^4$, when $c = 1 + 3i$, with the direction of $\arg(c)/2$, given by the quadratic part of the symbol—see Figure 1 on the next page.

Acknowledgement. I am grateful to Maciej Zworski for fruitful discussions on the subject of the present note. I would also like to thank Caroline Lasser for helpful discussions and advice, and Johannes Sjöstrand for stimulating comments. The partial support of the National Science Foundation under grant DMS-0304970 is gratefully acknowledged.

2 Proof of Theorem 1.1

Throughout this section, we shall assume, as we may, that $z_0 = 0$. Our starting point is the following essentially well-known result.

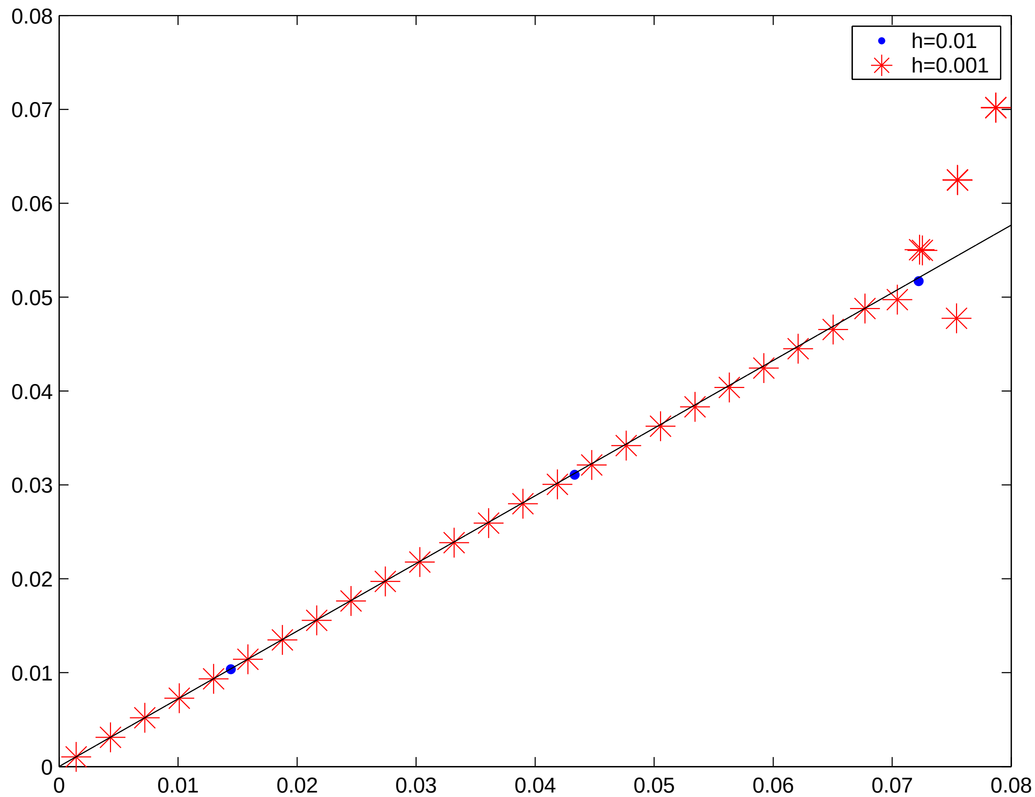

Figure 1: Numerical computation of the eigenvalues of $P = (hD_x)^2 + cx^2 + x^4$, when $c = 1 + 3i$, for two different values of h. When computing the eigenvalues of P, following [14], we discretized the operator using the Chebyshev spectral method. To compare the results of the computation with Theorem 1.1 we have also plotted the black solid line of the slope $\tan(\arg(c)/2) \approx 0.7208$.

Proposition 2.1 *Let $q(x,\xi)$ be a complex-valued quadratic form on $\mathbf{R}^2$ such that $q(x,\xi)=0$ precisely when $(x,\xi)=(0,0)$. Then the range of q on $\mathbf{R}^2$ is either all of $\mathbf{C}$ or there exists a proper closed convex cone $\subset \mathbf{C}$ which contains the range of q.*

Proof: This result is established in Lemma 3.1 of [10], and we shall recall the proof for completeness only. In suitable linear coordinates on $\mathbf{R}^2$, we may write for some $c\neq 0$,

$$q(x,\xi)=c(x-i\xi)(x-\lambda\xi),\quad \operatorname{Im}\lambda\neq 0.$$

Writing $z=x+i\xi$, we find that in the case when $\operatorname{Im}\lambda<0$,

$$q=\frac{1}{\alpha}\overline{z}(z+\gamma\overline{z}),\quad \alpha=\frac{2i}{c(i-\lambda)},$$

where $\gamma=(i+\lambda)(i-\lambda)^{-1}$ satisfies $|\gamma|<1$. Therefore,

$$\operatorname{Re}(\alpha q)=\operatorname{Re}(|z|^2+\gamma(\overline{z})^2)=\operatorname{Re}\left(|z|^2\left(1+\gamma\frac{(\overline{z})^2}{|z|^2}\right)\right)>0,$$

when $z\neq 0$, and in this case the range of q is contained in a closed angle of opening $<\pi$. In the opposite case, $\operatorname{Im}\lambda>0$, we find that with some $\beta\neq 0$,

$$\beta q=\gamma\,|z|^2+(\overline{z})^2,\quad |\gamma|<1.$$

It follows that the argument variation of βq along the circle $|z|=1$ is non-vanishing, and hence the range of q on $\mathbf{R}^2$ is all of $\mathbf{C}$. $\square$

Remark. According to Theorem 2.1.18 in [6], the range of a general complex-valued quadratic form on $\mathbf{R}^2$, restricted to $|z|=1$, is an ellipse, possibly degenerated to an interval or a point.

Applying Proposition 2.1 to the quadratic form $p_2(X)=\langle p''(0,0)X,X\rangle/2$, which begins the Taylor expansion of p at the origin, and using the exterior cone condition (1.7), we conclude that since 0 is a boundary point of the pseudospectrum of P, the range of p_2 is not all of $\mathbf{C}$, for otherwise, the argument variation of p along the positively oriented boundary of every small disc centered at 0, would be non-vanishing. We then know that for some $\alpha\in\mathbf{C}$, $|\alpha|=1$, the real part of αp_2 is positive definite. After a multiplication of P by α, we may and will assume henceforth that $\alpha=1$, so that the range of p_2 on $\mathbf{R}^2$ is contained in a set of the form

$$\Gamma=\{w\in\mathbf{C};|\operatorname{Im}w|\leq C\operatorname{Re}w\},$$

for some $C>0$.

By polarizaton, $p_2(X)$ gives rise to a symmetric bilinear form on $\mathbf{C}^2$, and then we write

$$p_2(X,Y) = \sigma(X, FY), \quad X, Y \in \mathbf{C}^2,$$

where σ is the complex symplectic form on $\mathbf{C}^2$, and $F : \mathbf{C}^2 \to \mathbf{C}^2$ is the Hamilton map of p_2—see Section 21.5 of [5] and also [7], for a systematic discussion of quadratic forms on symplectic vector spaces. Since $\mathrm{Re}\, p_2$ is positive definite on $\mathbf{R}^2$, it is true that F is bijective, and it is well known and easily seen that if μ is an eigenvalue of F then so is $-\mu$. When X is an eigenvector of F corresponding to the eigenvalue μ, we write

$$p_2(\overline{X}, X) = \sigma(\overline{X}, \mu X) = 2i\mu\sigma(\mathrm{Re}\, X, \mathrm{Im}\, X).$$

Here the left hand side is equal to $p_2(\mathrm{Re}\, X, \mathrm{Re}\, X) + p_2(\mathrm{Im}\, X, \mathrm{Im}\, X)$, and then we see that either $\mu/i \in \Gamma$ or $-\mu/i \in \Gamma$. In what follows we shall let μ stand for the unique eigenvalue of F for which $\mathrm{Re}\,(\mu/i) > 0$. Our goal is to show that as $h \to 0$, the spectrum of P near 0 accumulates towards the origin in the direction given by the argument of μ/i.

Remark. In Theorem 3.5 of [10], the spectrum of the Weyl quantization of a general complex valued elliptic quadratic form was computed. When specialized to the case at hand, the results of [10] show that the spectrum of $p_2^w(x, hD)$ is of the form $\{-i\mu h(2k+1),\, k = 0, 1, 2, \ldots\}$. Our Theorem 1.1 should therefore be viewed as an extension of the one-dimensional version of Theorem 3.5 of [10] to the case of a general analytic h-pseudodifferential operator. The proof of Theorem 3.5 in [10] employs complex linear canonical transformations, in order to essentially reduce the quadratic form globally to a complex multiple of the harmonic oscillator. In our case, we shall proceed similarly, but now merely microlocally, working near the critical point $(0,0)$ of the leading symbol of P.

We shall first use real canonical transformations to simplify p_2. After a real linear symplectic change of coordinates, and a conjugation of P by means of a corresponding unitary metaplectic operator, we may assume that as $(x,\xi) \to (0,0)$,

$$p(x,\xi) = p_2(x,\xi) + \mathcal{O}((x,\xi)^3) = \frac{\lambda}{2}(x^2+\xi^2) + \frac{i}{2}(ax^2 + b\xi^2 + 2cx\xi) + \mathcal{O}((x,\xi)^3), \quad \lambda > 0, \tag{2.1}$$

for some $a, b, c \in \mathbf{R}$. Here the quadratic form $\mathrm{Im}\, p_2$ can be diagonalized by means of an orthogonal transformation, which may be assumed to be orientation preserving. Hence this transformation is symplectic, since we are in dimension one, and we conclude that after an additional real linear symplectic transformation and a

metaplectic conjugation, we may reduce the quadratic part of p to the form in (2.1) with $c = 0$. We then write

$$p(x, \xi) = \frac{(\lambda + ia)}{2} \left(x^2 + d\xi^2\right) + \mathcal{O}((x, \xi)^3),$$

where $d = (\lambda + ia)^{-1}(\lambda + ib)$, and it is no restriction to assume that $\operatorname{Im} d \geq 0$. After an additional real symplectic dilation in (x, ξ), the symbol $p(x, \xi)$ takes the form

$$p(x, \xi) = \frac{(\lambda + ia)\,|d|^{1/2}}{2} \left(x^2 + e^{i\alpha}\xi^2\right) + \mathcal{O}((x, \xi)^3), \quad \alpha = \arg d \in [0, \pi). \tag{2.2}$$

Assuming that $\alpha > 0$, we shall now look for a small IR-deformation Λ of $\mathbf{R}^2$, such that the restriction of the quadratic part of p to Λ will have a constant argument, in a neighborhood of $(0, 0)$. We first argue formally, and consider $\Lambda_0 = T^*(e^{i\theta}\mathbf{R})$, where $\theta \in (0, \pi/4)$ is to be chosen. Notice that with $G(x, \xi) = \theta x\xi$, we can write $\Lambda_0 = \exp(iH_G)(\mathbf{R}^2)$. If $(x, \xi) \in \Lambda_0$ then $x = e^{i\theta}y$, $\xi = e^{-i\theta}\eta$ for $(y, \eta) \in \mathbf{R}^2$, and using (y, η) as real symplectic coordinates on Λ_0, we get

$$p|_{\Lambda_0} = p(\exp(iH_G)(y, \eta)) = \frac{(\lambda + ia)\,|d|^{1/2}}{2} \left(e^{2i\theta}y^2 + e^{i\alpha}e^{-2i\theta}\eta^2\right) + \mathcal{O}((y, \eta)^3).$$

We see that if we choose $\theta = \alpha/4$, then

$$\begin{aligned} p(\exp(iH_G)(y, \eta)) &= \frac{(\lambda + ia)d^{1/2}}{2} \left(y^2 + \eta^2\right) + \mathcal{O}((y, \eta)^3) \\ &= \frac{((\lambda + ia)(\lambda + ib))^{1/2}}{2}(y^2 + \eta^2) + \mathcal{O}((y, \eta)^3). \end{aligned}$$

Here we take the square root with the positive real part. Using the symplectic invariance of the Hamilton map F of p_2, we then check that here the coefficient

$$((\lambda + ia)(\lambda + ib))^{1/2}/2$$

is equal to μ/i, where μ is the eigenvalue of F, for which $\operatorname{Re}(\mu/i) > 0$.

We shall now introduce a global IR-manifold Λ which in a very small complex neighborhood of $(0, 0)$ agrees with $T^*(e^{i\theta}\mathbf{R})$, and further away from that set it is equal to $\mathbf{R}^2$. The manifold Λ will also be chosen so that the restriction of p to $\Lambda\backslash\{(0, 0)\}$ is non-vanishing. We remark that the following construction of Λ is similar to some arguments of [8] and [12] in the theory of resonances.

It will be convenient to work with the following FBI–Bargmann transform,

$$Tu(x) = Ch^{-3/4} \int e^{i\varphi(x,y)/h} u(y)\, dy, \quad x \in \mathbf{C}, \quad C > 0, \tag{2.3}$$

where

$$\varphi(x,y) = \frac{i}{e^{i\alpha/2}}\left(\frac{x^2}{2} + \frac{y^2}{2} - \sqrt{2}yx\right).$$

Notice that $\operatorname{Im}\varphi''_{yy} > 0$ and $\varphi''_{xy} \neq 0$, so that $\varphi(x,y)$ is an admissible phase in (2.3). Associated to T, there is a complex linear canonical transformation $\kappa_T : (y, -\varphi'_y(x,y)) \mapsto (x, \varphi_x(x,y))$, given by

$$\kappa_T(y,\eta) = \frac{1}{\sqrt{2}}\left(y - ie^{i\alpha/2}\eta, -ie^{-i\alpha/2}y + \eta\right), \quad (y,\eta) \in \mathbf{C}^2.$$

From the general theory [11], we know that κ_T maps $\mathbf{R}^2$ bijectively onto

$$\Lambda_{\Phi_0} := \left\{\left(x, \frac{2}{i}\partial_x \Phi_0(x)\right), \, x \in \mathbf{C}\right\},$$

where Φ_0 is the strictly subharmonic quadratic form given by

$$\Phi_0(x) = \sup_{y\in\mathbf{R}} -\operatorname{Im}\varphi(x,y).$$

A straightforward computation shows that

$$\Phi_0(x) = \cos(\alpha/2)\frac{(\operatorname{Re}x)^2}{2} + \frac{1+\sin^2(\alpha/2)}{\cos(\alpha/2)}\frac{(\operatorname{Im}x)^2}{2} + \sin(\alpha/2)\operatorname{Re}x\operatorname{Im}x. \tag{2.4}$$

We remark that Φ_0 is even strictly convex. Another computation shows next that the image of $T^*(e^{i\theta}\mathbf{R})$, $\theta = \alpha/4$, under κ_T is of the form Λ_{Φ_1}, where $\Phi_1(x) = \frac{1}{2}|x|^2$.

When $\epsilon > 0$, we take $\chi_\epsilon \in C_0^\infty(\mathbf{C})$, $0 \leq \chi \leq 1$, such that $\chi_\epsilon = 1$ when $|x| < e^{1/\epsilon}$, $\chi_\epsilon = 0$ for $|x| > e^{2/\epsilon}$, and such that

$$\partial^\alpha \chi_\epsilon(x) = \mathcal{O}_\alpha(\epsilon)\,|x|^{-|\alpha|}, \quad |\alpha| > 0.$$

Then the smooth function

$$\widetilde{\Phi} = \chi_\epsilon\Phi_1 + (1-\chi_\epsilon)\Phi_0$$

is strictly convex for $\epsilon > 0$ small enough, and satisfies

$$\widetilde{\Phi}(x) = \Phi_1(x), \quad |x| \leq e^{1/\epsilon},$$

$$\widetilde{\Phi}(x) = \Phi_0(x), \quad |x| \geq e^{2/\epsilon}.$$

In order to decrease the size of the neighborhood of 0 where $\widetilde{\Phi} \neq \Phi_0$, we introduce the strictly convex function

$$\Phi_\eta(x) = \left(\frac{\eta}{K_\epsilon}\right)^2 \widetilde{\Phi}\left(\frac{K_\epsilon x}{\eta}\right), \quad K_\epsilon := e^{2/\epsilon}, \quad 0 < \eta \ll 1, \tag{2.5}$$

so that $\Phi_\eta(x) = \Phi_0(x)$ for $|x| \geq \eta$. Notice that taking η sufficiently small, we may achieve that $\Phi_\eta - \Phi_0$ is arbitrarily small in C^1–norm. We then introduce the IR-manifold

$$\Lambda_{\Phi_\eta} : \xi = \frac{2}{i}\partial_x \Phi_\eta,$$

and remark that along Λ_{Φ_η}, we have

$$\xi = \xi(x) = \chi_\epsilon\left(\frac{K_\epsilon x}{\eta}\right)\frac{2}{i}\frac{\partial \Phi_1(x)}{\partial x} + \left(1 - \chi_\epsilon\left(\frac{K_\epsilon x}{\eta}\right)\right)\frac{2}{i}\frac{\partial \Phi_0(x)}{\partial x} + \mathcal{O}(\epsilon\,|x|),$$

uniformly in η. We claim now that with $\widetilde{p} = p \circ \kappa_T^{-1}$,

$$|\widetilde{p}(x, \xi(x)| \geq \frac{1}{\widetilde{C}}\,|x|^2, \quad \widetilde{C} > 0, \tag{2.6}$$

if $x \in \mathrm{neigh}(0, \mathbf{C})$ and $\epsilon > 0$ is small enough. Indeed, using (2.2) and computing the inverse of κ_T, we find that

$$\widetilde{p}(x, \xi) = 2((\lambda + ia)(\lambda + ib))^{1/2} ix\xi + \mathcal{O}((x, \xi)^3).$$

Using the strict convexity of the quadratic forms Φ_0 and Φ_1, we get

$$\begin{aligned}
\mathrm{Re}\,(ix\xi(x)) &= \chi_\epsilon\,|x|^2 + (1 - \chi_\epsilon)\mathrm{Re}\,(2x\partial_x \Phi_0) + \mathcal{O}(\epsilon\,|x|^2) \\
&= \chi_\epsilon\,|x|^2 + (1 - \chi_\epsilon)\langle x, \nabla \Phi_0(x)\rangle + \mathcal{O}(\epsilon\,|x|^2) \sim |x|^2,
\end{aligned}$$

uniformly in ϵ and η, and the transversal ellipticity property (2.6) follows at once. We conclude that for every fixed small $\eta > 0$, the IR-manifold $\Lambda := (\kappa_T)^{-1}(\Lambda_{\Phi_\eta})$ is equal to $\mathbf{R}^2$ away from an arbitrarily small, previously given neighborhood of $(0, 0)$ and it agrees with $T^*(e^{i\theta}\mathbf{R})$ near $(0, 0)$. Moreover, it is clear that the restriction of p to Λ vanishes precisely at $(0, 0)$.

Let us now recall the h-dependent Hilbert space $H(\Lambda) \simeq L^2(\mathbf{R})$, associated with the IR-manifold Λ and equipped with the norm obtained by replacing the weight Φ_0 by $\Phi = \Phi_\eta$ on the FBI transform side. We also introduce the Sobolev space $H(\Lambda, m)$ which agrees with $H(m)$ as a set, and whose norm is obtained from the

$H(m)$–norm by the same modification of the weight. Deforming the contour in the integral representation of P on the transform side, as explained in [9], we see that

$$P = \mathcal{O}(1) : H(\Lambda, m) \to H(\Lambda). \tag{2.7}$$

We summarize the discussion above in the following proposition.

Proposition 2.2 *Let P be as in the introduction and recall that in suitable real symplectic coordinates, the leading symbol p of P has the form* (2.2)*. There exists an IR-manifold $\Lambda \subset \mathbf{C}^2$, which coincides with $\Lambda_0 = T^*(e^{i\theta}\mathbf{R})$, $\theta = \alpha/4$, near $(0,0)$ and agrees with $\mathbf{R}^2$ outside an arbitrarily small neighborhood of $(0,0)$, such that $p \neq 0$ along $\Lambda \backslash \{(0,0)\}$. After applying the canonical transformation κ_T, associated with the transform* (2.3)*, so that $\mathbf{R}^2$ becomes Λ_{Φ_0}, with Φ_0 defined in* (2.4)*, Λ becomes Λ_Φ, where Φ is a strictly convex smooth function defined in* (2.5)*. For $x \in \mathrm{neigh}(0, \mathbf{C})$, we have*

$$\left| p \circ \kappa_T^{-1} \left(x, \frac{2}{i} \frac{\partial \Phi}{\partial x} \right) \right| \sim |x|^2 .$$

There exists a Fourier integral operator

$$U = e^{-G(x, hD_x)/h} = \mathcal{O}(1) : H(\Lambda) \to L^2(\mathbf{R}),$$

microlocally unitary near $(0,0)$, such that with P denoting the operator in (2.7)*, we have*

$$UP = \widetilde{P} U,$$

microlocally near $(0,0)$. Here $\widetilde{P} : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ has the leading symbol

$$p(\exp{(iH_G)}(x,\xi)) = \widetilde{p}(x,\xi) = \frac{\mu}{i}\left(x^2 + \xi^2\right) + \mathcal{O}((x,\xi)^3), \quad (x,\xi) \to (0,0),$$

where μ is the eigenvalue of the Hamilton map of the quadratic part of p at $(0,0)$, for which $\mathrm{Im}\, \mu > 0$.

In what follows we shall be interested in eigenvalues E of P with $|E| = \mathcal{O}(\epsilon^2)$, when $0 < \epsilon \ll 1$ is sufficiently small but fixed. The eigenfunctions corresponding to such eigenvalues are concentrated in a region where $|(x,\xi)| = \mathcal{O}(\epsilon)$, and therefore we introduce a change of variables $x = \epsilon y$. Then

$$\frac{1}{\epsilon^2} P(x, hD_x; h) = \frac{1}{\epsilon^2} P(\epsilon(y, \widetilde{h} D_y); h), \quad \widetilde{h} = \frac{h}{\epsilon^2}.$$

When viewed as an $\widetilde{h}$-pseudodifferential operator, $\epsilon^{-2}P$ has a corresponding new symbol

$$\frac{1}{\epsilon^2}P(\epsilon(y,\eta)) \sim \frac{1}{\epsilon^2}p(\epsilon(y,\eta)) + \widetilde{h}p_1(\epsilon(y,\eta)) + \dots,$$

to be considered in a region where $|(y,\eta)| = \mathcal{O}(1)$.

This scaling reduction together with Proposition 2.2 allows us to reduce the further analysis to an analytic $\widetilde{h}$-pseudodifferential operator P_ϵ, microlocally defined near $(0,0) \in \mathbf{R}^2$, whose (not necessarily real) leading symbol has the form

$$p_\epsilon(x,\xi) := p_{0,\epsilon}(x,\xi) = \frac{x^2+\xi^2}{2} + \epsilon r_\epsilon(x,\xi), \quad r_\epsilon(x,\xi) = \mathcal{O}((x,\xi)^3).$$

The full symbol is

$$P_\epsilon(x,\xi;\widetilde{h}) \sim \sum_{j=0}^{\infty} \widetilde{h}^j p_{j,\epsilon}(x,\xi),$$

with $p_{j,\epsilon}(x,\xi)$ holomorphic in a fixed complex neighborhood of $(x,\xi) = (0,0)$.

From Section 5 of [4] we recall that for every small enough ϵ, there exists a holomorphic canonical transformation

$$\kappa_\epsilon : \mathrm{neigh}((0,0),\mathbf{C}^2) \to \mathrm{neigh}((0,0),\mathbf{C}^2),$$

such that $\kappa_\epsilon(0,0) = (0,0)$, and $\operatorname{Im}\kappa_\epsilon(x,\xi) = \mathcal{O}(\epsilon)$ when x,ξ are real, and such that

$$p_\epsilon \circ \kappa_\epsilon(x,\xi) = g_\epsilon\left(\frac{x^2+\xi^2}{2}\right).$$

Here $g_\epsilon(E)$ is an analytic function of E, with $g_\epsilon(0) = 0$, and such that $\operatorname{Im} g_\epsilon(E) = \mathcal{O}(\epsilon)$ for real E, and $\operatorname{Re} g_\epsilon'(0) > 0$. We may also remark that we have in fact $g_\epsilon(E) = E + \mathcal{O}(\epsilon E)$.

Associated to the canonical transformation κ_ϵ, we introduce a global IR-manifold $\Lambda_\epsilon \subset \mathbf{C}^2$, which is ϵ–close to $\mathbf{R}^2$, equals $\kappa_\epsilon(\mathbf{R}^2)$ in a complex neighborhood of $(0,0)$, and agrees with the real phase space further away from this set. Precisely as in [4], we then implement κ_ϵ by means of an elliptic $\widetilde{h}$–Fourier integral operator U_ϵ,

$$U_\epsilon = \mathcal{O}(1) : L^2(\mathbf{R}) \to H(\Lambda_\epsilon).$$

Then the action of P_ϵ on $H(\Lambda_\epsilon)$ is, microlocally near $(0,0)$, unitarily equivalent to the $\widetilde{h}$-pseudodifferential operator

$$\widetilde{P}_\epsilon = U_\epsilon^{-1} P_\epsilon U_\epsilon : L^2(\mathbf{R}) \to L^2(\mathbf{R}),$$

whose complete Weyl symbol has the form

$$\widetilde{P}_\epsilon(x,\xi;\widetilde{h}) \sim \sum_{j=0}^{\infty} \widetilde{h}^j \widetilde{p}_{j,\epsilon}(x,\xi),$$

with

$$\widetilde{p}_{0,\epsilon}(x,\xi) = g_\epsilon\left(\frac{x^2+\xi^2}{2}\right).$$

Continuing to follow [4] (where also further references are given), let us now recall how to simplify the lower order terms in an operator whose principal symbol is modelled on the one-dimensional harmonic oscillator. In doing so, we consider a formal h-pseudodifferential operator $Q(x, hD_x; h)$, defined microlocally near $(0,0) \in \mathbf{R}^2$, with symbol

$$Q(x,\xi;h) \sim q_0(x,\xi) + hq_1(x,\xi) + \ldots,$$

where q_j are holomorphic in a fixed complex neighborhood of $(x,\xi) = (0,0)$, and

$$q_0(x,\xi) = g_0((x^2+\xi^2)/2),$$

with g_0 holomorphic near 0, $g_0(0) = 0$, $g_0'(0) \neq 0$. In Section 5 of [4] we recalled that by means of an averaging procedure we can construct

$$A(x,\xi;h) \sim a_0(x,\xi) + ha_1(x,\xi) + \ldots,$$

with all a_j holomorphic in a j-independent neighborhood of $(0,0)$, such that

$$e^{iA(x,hD_x;h)} Q(x,hD_x;h) e^{-iA(x,hD_x;h)} = g\left(\frac{x^2+(hD_x)^2}{2};h\right). \tag{2.8}$$

Here

$$g(E;h) \sim \sum_{j=0}^{\infty} g_j(E) h^j,$$

with g_0 as above, and $g_j(E)$ holomorphic in a fixed neighborhood of $E = 0$. As in [4], when deriving (2.8) we make use of the fact that if f is a smooth (holomorphic) function on $\mathbf{R}^2$ which depends on $(x^2+\xi^2)/2$ only, then $f^w(x,hD_x) = \widetilde{f}\left(\frac{1}{2}(x^2+(hD_x)^2)\right)$, for some $\widetilde{f}$ with $\widetilde{f} = f + \mathcal{O}(h)$—see [7] for this essentially well-known result.

Applying this discussion to $\widetilde{P}_\epsilon$, we conclude that there exists

$$A_\epsilon(x,\xi;\widetilde{h}) \sim A_{0,\epsilon}(x,\xi) + \widetilde{h} A_{1,\epsilon}(x,\xi) + \ldots,$$

with each $A_{j,\epsilon}$ holomorphic in a fixed neighborhood of $(0,0)$ in $\mathbf{C}^2$, such that after a conjugation by $e^{iA_\epsilon(x,\widetilde{h}D_x;\widetilde{h})}$, the operator $\widetilde{P}_\epsilon$ becomes

$$\widehat{P}_\epsilon = G_\epsilon\left(\frac{x^2+(\widetilde{h}D_x)^2}{2};\widetilde{h}\right),$$

where

$$G_\epsilon(q;\widetilde{h}) \sim \sum_{j=0}^{\infty} \widetilde{h}^j G_{j,\epsilon}(q),$$

with all $G_{j,\epsilon}$ holomorphic in q in a j-independent neighborhood of $0 \in \mathbf{C}$. Moreover, $G_{0,\epsilon} = g_\epsilon$ above.

Combining Proposition 2.2 together with the reductions above, we arrive at the following result.

Proposition 2.3 *Let P and p be as in the introduction and recall that we assume that the real part of the Hessian of p at $(x,\xi)=(0,0)$ is positive definite. Let $\epsilon > 0$ be small enough but fixed, and let us view the operator*

$$\frac{1}{\epsilon^2}P(x,hD_x) = \frac{1}{\epsilon^2}P\left(\epsilon(y,\widetilde{h}D_y)\right), \quad x=\epsilon y, \quad \widetilde{h}=\frac{h}{\epsilon^2},$$

as an $\widetilde{h}$-pseudodifferential operator, with the leading symbol $p_\epsilon(y,\eta) = \frac{1}{\epsilon^2}p(\epsilon(y,\eta))$. There exists an IR-manifold $\widehat{\Lambda} \subset \mathbf{C}^2$, with $(0,0)\in\widehat{\Lambda}$, which coincides with $\mathbf{R}^2$ away from a neighborhood of $(0,0)$ and is close to $\mathbf{R}^2$ everywhere, such that the restriction of p_ϵ to $\widehat{\Lambda}$ is elliptic outside an arbitrarily small neighborhood of $(0,0)$. Furthermore, there exists an elliptic Fourier integral operator

$$U : H(\widehat{\Lambda}) \to L^2(\mathbf{R}),$$

such that, microlocally near $(0,0)\in\widehat{\Lambda}$,

$$U\left(\frac{1}{\epsilon^2}P\right) = \widehat{P}_\epsilon U.$$

Here $\widehat{P}_\epsilon = \widehat{P}_\epsilon\left((1/2)((\widetilde{h}D_x)^2+x^2);\widetilde{h}\right)$ has the complete Weyl symbol,

$$\widehat{P}_\epsilon(x,\xi;\widetilde{h}) \sim \sum_{j=0}^{\infty}\widetilde{h}^j G_{j,\epsilon}\left(\frac{x^2+\xi^2}{2}\right),$$

with the functions $G_{j,\epsilon}$ holomorphic in a common neighborhood of $0 \in \mathbf{C}$. We have $G_{0,\epsilon}(0) = 0$ and

$$\arg\left(G'_{0,\epsilon}(0)\right) = \arg\left(\frac{\mu}{i}\right) + \mathcal{O}(\epsilon),$$

where μ is the unique eigenvalue of the Hamilton map of the Hessian of p at $(0,0)$, for which $\operatorname{Im}\mu > 0$.

With Proposition 2.3 available, we are now in a position to solve a suitable Grushin problem in the globally defined Hilbert space $H(\widehat{\Lambda})$, and it is clear that the setup of the relevant Grushin problem is exactly the same as in [4]. (See also [13] for a recent general presentation of Grushin problem techniques in semiclassical analysis and spectral theory.) The analysis of the Grushin problem in question is even simplified when compared with [4], as we can use ellipticity along $\widehat{\Lambda}$, when away from a neighborhood of $(0,0)$, while near $(0,0)$, the operator is reduced microlocally to a function of the harmonic oscillator. In conclusion, we obtain the following result.

Proposition 2.4 *Let $\epsilon > 0$ be small enough but independent of h. The eigenvalues of P in an open disc centered at 0 with radius $\mathcal{O}(\epsilon^2)$ are of the form*

$$\sim \sum_{j=0}^{\infty} h^j \epsilon^{2-2j} G_{j,\epsilon}\left(\frac{h}{\epsilon^2}\left(k + \frac{1}{2}\right)\right), \quad k \in \mathbf{N},$$

with $G_{j,\epsilon}(q)$ holomorphic in $q \in \operatorname{neigh}(0,\mathbf{C})$. We have

$$G_{0,\epsilon}(q) = \frac{\mu}{i} q (1 + \mathcal{O}(\epsilon)).$$

Combining Proposition 2.4 together with a scaling argument as in [9], [12], and [4], we obtain Theorem 1.1.

References

[1] E. B. Davies, *Pseudo-spectra, the harmonic oscillator, and complex resonances*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **455** (1999), 585–599.

[2] E. B. Davies, *Semi-classical states for non-self-adjoint Schrödinger operators*, Comm. Math. Phys. **200** (1999), 35–41.

[3] N. Dencker, J. Sjöstrand, and M. Zworski, *Pseudospectra of semiclassical (pseudo)-differential operators*, Comm. Pure Appl. Math. **57** (2004), 384–415.

[4] M. Hitrik and J. Sjöstrand, Non-selfadjoint perturbations of selfadjoint operators in 2 dimensions I, Ann. Henri Poincaré, to appear.

[5] L. Hörmander, *The Analysis of linear partial differential operators*, Vol. III, Springer Verlag, New York, 1985.

[6] L. Hörmander, *Notions of convexity*, Birkhäuser Boston, 1994.

[7] L. Hörmander, *Symplectic classification of quadratic forms, and general Mehler formulas*, Math. Z., **219** (1995), 413–449.

[8] N. Kaidi and P.Kerdelhué, *Forme normale de Birkhoff et résonances*, Asymp. Analysis, **23** (2000), 1–21.

[9] A. Melin and J. Sjöstrand, *Bohr-Sommerfeld quantization condition for non-selfadjoint operators in dimension 2*, Autour de l'analyse microlocale, Astérisque **284** (2003), 181-244.

[10] J. Sjöstrand, *Parametrices for pseudodifferential operators with multiple characteristics*, Arkiv för Matematik, **12** (1974), 85–130.

[11] J. Sjöstrand, *Function spaces associated to global I-Lagrangian manifolds*, Structure of solutions of differential equations, 369–423, World Sci. Publishing, River Edge, NJ, 1996.

[12] J. Sjöstrand, *Resonances associated to a closed hyperbolic trajectory in dimension 2*, preprint, September 2002.

[13] J. Sjöstrand and M. Zworski, *Elementary linear algebra for advanced spectral problems*, preprint, December 2003.

[14] L. N. Trefethen, *Spectral methods in Matlab*, SIAM, Philadelphia, 2000.

[15] M. Zworski, *A remark on a paper of E. B. Davies" "Semi-classical states for non-self-adjoint Schrödinger operators"*, Proc. Amer. Math. Soc. **129** (2001), 2955–2957.

[16] M. Zworski, *Numerical linear algebra and solvability of partial differential equations*, Comm. Math. Phys. **229** (2002), 293–307.